



The Regulator Problem with Robust Stability*

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The maximally achievable degree of robustness of stability, in the sense of coprime factor perturbations, is determined for linear servo systems.

Abstract—The design of a controller such that the closed-loop system will track reference signals or reject disturbance signals from a specified class is known as the 'servomechanism problem' or the 'regulator problem'. For the regulator problem to be solvable with robust closed-loop stability, the plant obviously needs to be such that the regulation problem and the robust stabilization problem are solvable separately. In this paper we determine the extra conditions that are necessary and sufficient for the two problems to be solved simultaneously. It turns out that these conditions can be given a simple geometric interpretation in terms of a multivariable version of the Nyquist curve of the plant.

1. INTRODUCTION

In classical control theory, perhaps the most central issue is the reconciliation of various design objectives. Modern control theory, on the other hand, has tended to isolate specific aspects of design and to provide separate solutions for the associated problems. While the modern approach has brought much progress, recent ('postmodern') research has emphasized the need for a study of the trade-offs between various design objectives in order to work towards a unification of the classical and the modern theory. Various approaches have been suggested, including realizability constraints (Freudenberg and Looze, 1988), optimization methods (Boyd and Barratt, 1991) and loop shaping (McFarlane and Glover 1990; Doyle *et al.*, 1992). In this paper we study the interaction between robust stability requirements and

regulation requirements. It turns out that this particular interaction can be described in a remarkably simple way.

By a *regulation requirement*, we understand in this paper a requirement on the closed-loop systems to reject or follow a signal produced by an 'exosystem' of the form $\dot{z} = Fz$, $d = Hz$, where the eigenvalues of the matrix F are located on the imaginary axis. Signals that can be described in this way includes steps, ramps, and sinusoids of fixed frequency. In particular, the rejection of constant disturbances under closed-loop stability is one of the most classical problems in control theory (Maxwell, 1868). The regulator problem has been extensively studied from various points of view during the 1970s and early 1980s; see the references in Wonham (1979) and Basile and Marro (1992).

The 1980s also saw new developments in the theory of *robust stabilization*. Among the nonparametric perturbation models, that based on normalized coprime factorizations drew considerable attention, especially after it was shown by Glover and McFarlane (1989) that the problem of designing an optimally robust controller with respect to this perturbation class has a relatively straightforward solution. We shall use the same perturbation model in this paper.

The main subject of the paper will be to combine the regulation requirement with the robust stability requirement (in the sense of coprime factor perturbations). A first concern is to express the two requirements in a common framework. For this, we use the formulation in terms of subspace-valued functions, which can be traced back to Martin and Hermann (1978) and Brockett and Byrnes (1981). It has already been demonstrated (Qiu and Davison 1992; Schumacher 1992) that subspace-valued functions are excellently suited to describe robust stability properties. In this paper we employ the same framework for the regulator problem. It has been shown by Cevik and Schumacher (1994) that the regulator problem can be formulated as

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an interpolation problem for subspace-valued functions. As a consequence of this, using finite-dimensional geometry, one readily obtains necessary conditions for the solvability of the regulator problem when a stability margin γ is imposed. We show that these conditions are also sufficient if two other (obvious) conditions are satisfied, namely that the regulation problem and the robust stabilization problem are solvable separately.

The paper is organized as follows. In the next section we give precise formulations of the problem we want to solve and recall the relevant results from Cevik and Schumacher (1994). The solution of the regulator problem with robust stability is given in Section 3, and is followed by an example in Section 4. Conclusions are stated in Section 5. In the Appendix we provide a detailed proof of one of the lemmas in the main text.

2. PROBLEM FORMULATION AND PRELIMINARIES

The regulator problem may be formulated as follows, assuming that the observed outputs coincide with the regulated outputs (cf. Basile and Marro (1992; p. 317), where the problem below is referred to as the autonomous regulator problem). Consider a finite-dimensional linear time-invariant system of the form

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t), \quad (1)$$

$$\dot{x}_2(t) = A_{22}x_2(t), \quad (2)$$

$$y(t) = C_1x_1(t) + C_2x_2(t). \quad (3)$$

The interpretation is as follows: x_1 denotes the state of the plant, whereas x_2 is the state of an 'exosystem' that generates signals which can be disturbances or references. The matrix A_{22} has its eigenvalues on the imaginary axis, allowing the reference/disturbance signals to be steps, ramps, sinusoids etc. The variable $y(t)$ should converge to zero, irrespective of the presence of the signals generated by the exosystem. This is to be achieved by a linear time-invariant compensator of the form

$$\dot{z}(t) = Fz(t) + Gy(t), \quad (4)$$

$$u(t) = Hz(t) + Jy(t). \quad (5)$$

We shall consider the regulator problem under the following standing assumptions.

Assumptions. The system (1)–(3) satisfies the following:

(A1) the pair (A_{11}, B_1) is stabilizable;

(A2) the pair (C, A) given by

$$C = [C_1 \ C_2], \quad A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad (6)$$

is detectable;

(A3) all eigenvalues of A_{22} are on the imaginary axis;

(A4) for every eigenvalue λ of A_{22} , the matrix

$$\begin{bmatrix} \lambda I - A_{11} & -B_1 \\ C_1 & 0 \end{bmatrix}$$

has full column rank.

Assumption (A1) is necessary for the plant to be stabilizable by a feedback compensator, and so this is a natural assumption to make. Detectability of the pair (C_1, A_{11}) is necessary as well for closed-loop stability to be achieved by a compensator of the form (4), (5); assumption (A2) requires a bit more, however. It can be argued that (A2) may be assumed without essential loss of generality in the regulator problem (cf. Wonham, 1979; § 8.1). Instead of (A3), the usual assumption is that the exosystem poles are in the closed right half-plane (cf. e.g. Francis, 1977); although (A3) is of course stronger, it hardly represents a restriction from the applications point of view. Assumption (A4) is not quite so harmless because it implies that the number of outputs is at least equal to the number of inputs, whereas it is well known (Wonham, 1979, Chap 8) that the regulator problem can only be 'well-posed' if the number of outputs is at most equal to the number of inputs. One may therefore say that (A4) essentially limits one to the case in which the number of control inputs is equal to the number of regulated outputs. The assumption requires that the plant zeros do not coincide with the exosystem poles, which is a well-known condition in connection with the regulator problem (Wonham, 1979, Theorem 8.3; Basile and Marro, 1992, Corollary 5.2-2).

An important role in our analysis will be played by certain subspace-valued functions associated with plant and controller. With the plant given by the triple (A_{11}, B_1, C_1) we associate the function

$$\mathcal{P}(s) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \middle| \exists x \text{ s.t.} \right. \\ \left. \begin{bmatrix} sI - A_{11} & 0 & -B_1 \\ C_1 & -I & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ u \end{bmatrix} = 0 \right\}, \\ \mathcal{P}(\infty) = \text{im} \begin{bmatrix} 0 \\ I \end{bmatrix}. \quad (7)$$

With the full system (1)–(3), we associate

$$\mathcal{M}(s) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \middle| \exists x_1, x_2 \text{ s.t.} \right. \\ \left. \begin{bmatrix} sI - A_{11} & -A_{12} & 0 & -B_1 \\ 0 & sI - A_{22} & 0 & 0 \\ C_1 & C_2 & -I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y \\ u \end{bmatrix} = 0 \right\}, \quad (8)$$

$$\mathcal{M}(\infty) = \text{im} \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

In the same way, we finally associate with the controller the subspace-valued function

$$\mathcal{C}(s) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \middle| \exists z \text{ s.t.} \begin{bmatrix} sI - F & -G & 0 \\ H & J & -I \end{bmatrix} \right. \\ \left. \times \begin{bmatrix} z \\ y \\ u \end{bmatrix} = 0 \right\}, \quad \mathcal{C}(\infty) = \text{im} \begin{bmatrix} I \\ J \end{bmatrix}. \quad (9)$$

Note that all functions take values in the set of subspaces of the product space $\mathcal{Y} \times \mathcal{U}$, which is an $(m + p)$ -dimensional space if m is the number of inputs and p is the number of outputs. The functions above may be considered as functions on the extended complex plane $\mathbb{C} \cup \{\infty\}$, but we shall only need their values on the closed right half-plane

$$\mathbb{C}^+ \stackrel{\text{def}}{=} \{s \in \mathbb{C} \mid \text{Re } s \geq 0\} \cup \{\infty\}. \quad (10)$$

The closed-loop system takes the form

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix} (t) = A_e \begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix} (t), \quad (11)$$

$$y(t) = [C_1 \quad 0 \quad C_2] \begin{bmatrix} x_1 \\ z \\ x_2 \end{bmatrix} (t), \quad (12)$$

where

$$A_e = \begin{bmatrix} A_{11} + B_1 J C_1 & B_1 H & A_{12} + B_1 J C_2 \\ G C_1 & F & G C_2 \\ 0 & 0 & A_{22} \end{bmatrix}. \quad (13)$$

The compensator is said to satisfy the *internal*

stability requirement if the closed-loop system is stable when $x_2(t) = 0$, that is, if the matrix

$$\begin{bmatrix} A_{11} + B_1 J C_1 & B_1 H \\ G C_1 & F \end{bmatrix}$$

is stable.

In order to define a requirement for *robust stability*, it is of interest to consider an equivalent formulation based on the subspace-valued functions (7) and (9), and on the notion of the minimal angle between subspaces. The minimal angle between two subspaces \mathcal{Y} and \mathcal{Z} of a unitary space \mathcal{X} is defined as follows (see e.g. Gohberg and Krein, 1969, p. 339):

$$\sin \phi(\mathcal{Y}, \mathcal{Z}) \\ = \min \{ \|y - z\| \mid y \in \mathcal{Y}, z \in \mathcal{Z}, \|y\| = 1 \}, \\ 0 \leq \phi \leq \frac{1}{2}\pi. \quad (14)$$

Note that the minimal angle is nonzero if and only if the two subspaces intersect only in 0. If this condition holds, another formula for the minimal angle is given by (see again e.g. Gohberg and Krein, 1969, p. 339)

$$\sin \phi(\mathcal{Y}, \mathcal{Z}) = \|\Pi_{\mathcal{Z}}^{\mathcal{Y}}\|^{-1}, \quad (15)$$

where $\Pi_{\mathcal{Z}}^{\mathcal{Y}}$ denotes the skew projection along \mathcal{Y} onto \mathcal{Z} , defined on $\mathcal{Y} + \mathcal{Z}$.

Lemma 2.1. The closed-loop system formed by the plant (A_{11}, B_1, C_1) and the compensator (4), (5) is stable if and only if

$$\min_{s \in \mathbb{C}^+} \sin \phi(\mathcal{P}(s), \mathcal{C}(s)) > 0. \quad (16)$$

Proof. First assume that the closed-loop system is stable. This implies (cf. Schumacher, 1992; Cevik and Schumacher, 1994) that $\sin \phi(\mathcal{P}(\infty), \mathcal{C}(\infty))$ is positive as well. It follows from Martin and Hermann (1978) (see also de Does and Schumacher, 1994b; Cevik and Schumacher, 1994) that the functions $s \mapsto \mathcal{P}(s)$ and $s \mapsto \mathcal{C}(s)$ are continuous mappings from \mathbb{C}^+ to the Grassmannian manifolds $G^m(\mathcal{Y} \times \mathcal{U})$ and $G^p(\mathcal{Y} \times \mathcal{U})$ respectively. (Recall that for a given finite-dimensional vector space \mathcal{X} , the Grassmannian manifold $G^k(\mathcal{X})$ is the set of all k -dimensional subspaces of \mathcal{X} , equipped with the gap topology—see e.g. Glazman and Ljubić (1974, Section IV.7). It is then seen from de Does and Schumacher (1994a, Lemma 2.4) that the function $s \mapsto \sin \phi(\mathcal{P}(s), \mathcal{C}(s))$ is continuous. Because \mathbb{C}^+ is compact, it follows that this function indeed assumes a minimum on \mathbb{C}^+ , which must be positive by the assumption of closed-loop stability. The converse is immediate for instance from Cevik and Schumacher (1994, Lemma 2.5). \square

which we shall call the *mingling* operator. It is defined by

$$Mi: (x_1, \dots, x_r, y_1, \dots, y_r) \mapsto (x_1, y_1, \dots, x_r, y_r). \tag{21}$$

We shall use the mingling operator between various spaces and even use its obvious generalization to products of more than two factors, employing the same symbol Mi every time; this relatively severe abuse of notation should cause no confusion.

In addition to the blow-ups of matrix functions, we shall also need blown-up versions of the various subspace-valued functions that were introduced above. For the functions $\mathcal{P}(s)$ and $\mathcal{C}(s)$ defined in (7) and (9) respectively, these can be defined via either image or kernel representations as follows:

$$\mathcal{P}^{[r]}(s) = \ker \tilde{P}^{[r]}(s) = \text{im } P^{[r]}(s), \tag{22}$$

$$\mathcal{C}^{[r]}(s) = \ker \tilde{C}^{[r]}(s) = \text{im } C^{[r]}(s). \tag{23}$$

It follows from Cevik and Schumacher (1994, Lemmas 3.3 and 3.4) that this definition is unambiguous. The subspace-valued function $\mathcal{M}(s)$ defined in (8) requires more care because it has singularities. Note that we may write

$$\mathcal{M}(s) = \Pi \ker \begin{bmatrix} sI - A & 0 & -B \\ C & -I & 0 \end{bmatrix}, \tag{24}$$

where

$$B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

and Π denotes the natural projection from $\mathcal{X} \times \mathcal{Y} \times \mathcal{U}$ to $\mathcal{Y} \times \mathcal{U}$. We now define $\mathcal{M}^{[r]}(s)$ by

$$\mathcal{M}^{[r]}(s) = \Pi^{[r]} \ker \begin{bmatrix} sI - A & 0 & -B \\ C & -I & 0 \end{bmatrix}^{[r]}, \tag{25}$$

$$\mathcal{M}^{[r]}(\infty) = \text{im} \begin{bmatrix} 0 \\ I \end{bmatrix}^{[r]}.$$

A matrix function $\tilde{M}(s)$ will be called a *kernel representation* of the *sequence* of subspace-valued functions $\mathcal{M}^{[r]}(s)$ if $\ker \tilde{M}^{[r]}(s) = \mathcal{M}^{[r]}(s)$ for all s in the considered domain. It has been shown by Cevik and Schumacher (1994, Lemma 3.9) that such representations do indeed exist.

For ease of notation, we introduce

$$\mathcal{X} = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \mid y = 0 \right\} \tag{26}$$

and denote the natural projection from $\mathcal{Y} \times \mathcal{U}$ to \mathcal{Y} by $\tilde{K} = [I \ 0]$, so that

$$\mathcal{X} = \ker \tilde{K} = \text{im} \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{27}$$

Regarding \tilde{K} as a constant matrix-valued

function, we can also consider $\tilde{K}^{[r]}$ which is simply a block-diagonal matrix with \tilde{K} on the diagonal entries, and $\mathcal{X}^{[r]} = \ker \tilde{K}^{[r]}$. By the *multiplicity* of an eigenvalue of a matrix we mean the length of the longest Jordan chain associated with that eigenvalue. The main result of Cevik and Schumacher (1994) can now be formulated as follows.

Theorem 2.2. A controller of the form (4), (5) is a solution to the regulator problem with internal stability if and only if the associated subspace-valued function $\mathcal{C}(s)$ satisfies

$$\mathcal{C}^{[r]}(\lambda) \cap \mathcal{M}^{[r]}(\lambda) \subset \mathcal{X}^{[r]} \quad \forall \lambda \text{ in } \sigma(A_{22}) \text{ of multiplicity } r \tag{28}$$

and

$$\mathcal{C}(\lambda) \oplus \mathcal{P}(\lambda) = \mathcal{Y} \times \mathcal{U} \quad \forall \lambda \in \mathbb{C}^+. \tag{29}$$

We shall need the following facts (cf. Cevik and Schumacher, 1994).

Lemma 2.3. Let \mathcal{W} be a vector space, and let \mathcal{C} , \mathcal{P} and \mathcal{M} be subspaces of \mathcal{W} such that $\mathcal{P} \oplus \mathcal{C} = \mathcal{W}$ and $\mathcal{P} \subset \mathcal{M}$. Denoting the projection onto \mathcal{C} along \mathcal{P} by $\Pi_{\mathcal{C}}^{\mathcal{P}}$, we have

$$\mathcal{C} \cap \mathcal{M} = \Pi_{\mathcal{C}}^{\mathcal{P}} \mathcal{M}. \tag{30}$$

Lemma 2.4. Consider the system (1)–(3), and assume that the pair (C_1, A_{11}) is detectable, and that all eigenvalues of A_{22} are in the closed right half-plane. Under these conditions, which are in particular satisfied if assumptions (A2) and (A3) hold, assumption (A4) holds if and only if

$$\mathcal{P}(\lambda) \cap \mathcal{X} = \{0\}. \tag{31}$$

We now prove the following.

Lemma 2.5. Let \mathcal{W} be a finite-dimensional vector space with given subspaces \mathcal{M} , \mathcal{P} and \mathcal{X} . Assume that $\mathcal{P} \subset \mathcal{M}$ and $\mathcal{P} \cap \mathcal{X} = \{0\}$. Under these conditions, there exists a subspace \mathcal{C} such that

$$\mathcal{C} \oplus \mathcal{P} = \mathcal{W}, \quad \mathcal{C} \cap \mathcal{M} \subset \mathcal{X} \tag{32}$$

if and only if $\mathcal{M} = \mathcal{P} + (\mathcal{X} \cap \mathcal{M})$.

Proof. First assume that there exists a subspace \mathcal{C} satisfying the stated conditions. We then have $\mathcal{M} = \mathcal{M} \cap (\mathcal{C} + \mathcal{P}) = (\mathcal{M} \cap \mathcal{C}) + \mathcal{P} \subset (\mathcal{X} \cap \mathcal{M}) + \mathcal{P}$, whereas the reverse inclusion is immediate from the assumption $\mathcal{P} \subset \mathcal{M}$. Now, assume that the condition $\mathcal{M} = \mathcal{P} + (\mathcal{X} \cap \mathcal{M})$ holds. Let \mathcal{T} be any complement of \mathcal{M} in \mathcal{W} , and take $\mathcal{C} = (\mathcal{X} \cap \mathcal{M}) \oplus \mathcal{T}$. We have to show that \mathcal{C} is complementary to \mathcal{P} , and that $\mathcal{C} \cap \mathcal{M} \subset \mathcal{X}$. The first claim is immediate by noting that the assumptions imply

Another way to express the above result is that $\mathcal{P}(s)$ and $\mathcal{C}(s)$ should be complementary at each point $s \in \mathbb{C}^+$. It has been shown by Schumacher (1992) that the minimal angle is the appropriate measure of the robustness of complementarity of two subspaces \mathcal{Y} and \mathcal{Z} , in the sense that it gives exactly the distance (in the sense of the gap) of \mathcal{Y} to the set of subspaces \mathcal{Y}' that are not complementary to \mathcal{Z} . As a measure of robustness of stability, we shall therefore take

$$\sin \phi(\mathcal{P}, \mathcal{C}) \stackrel{\text{def}}{=} \min_{s \in \mathbb{C}^+} \sin \phi(\mathcal{P}(s), \mathcal{C}(s)). \quad (17)$$

The minimum is actually achieved on the imaginary axis or at infinity. The above measure can also be motivated in other ways, and has been used for instance by Vidyasagar and Kimura (1986), Glover and McFarlane (1989), Georgiou and Smith (1990) and Vinnicombe (1993). As the notation suggests, the expression $\phi(\mathcal{P}, \mathcal{C})$ can be interpreted as an angle between linear spaces associated with plant and controller (Ober and Sefton, 1991; Schumacher, 1992).

Now consider the following problems.

Problem 1. (Regulator problem with internal stability: RPIS). Given the plant and exosystem (1)–(3), find a compensator of the form (4), (5) such that the closed-loop system (11)–(13) is internally stable and satisfies the *regulation requirement*

$$\mathcal{X}_+(A_e) \subset \ker [C_1 \ 0 \ C_2], \quad (18)$$

where $\mathcal{X}_+(A_e)$ denotes the unstable subspace of A_e .

Problem 2. (Robust stabilization problem with margin γ : RSP (γ)). Given the plant (1), (2) and γ with $0 < \gamma < 1$, find a compensator of the form (4), (5) that satisfies the *robust stability requirement*

$$\min_{s \in \mathbb{C}^+} \sin \phi(\mathcal{P}(s), \mathcal{C}(s)) > \gamma. \quad (19)$$

Problem 3. (Regulator problem with robust stability margin γ : RPRS (γ)). Given the plant and exosystem (1)–(3) and γ with $0 < \gamma < 1$, find a compensator of the form (4), (5) such that both the regulation property (18) and the robust stability property (19) hold.

Necessary and sufficient conditions for RPIS and RSP (γ) to be solvable, along with synthesis procedures to obtain a suitable compensator, are well known; for RPIS, see Wonham (1979) and

Basile and Marro (1992) and references therein, and for RSP (γ), see Vidyasagar and Kimura (1986), Glover and McFarlane (1989) and McFarlane and Glover (1990). One may refer in particular to Wonham (1979, Theorem 8.1) for RPIS and McFarlane and Glover (1990, Theorem 4.14) for RSP (γ). Our purpose in this paper is to get the same results for RPRS (γ). We shall do this by making extensive use of the results of Cevik and Schumacher (1994), which characterize the regulator problem as an interpolation problem. Before we can state the main theorem from that paper, we need the following definitions.

Consider an analytic function $M(s)$ defined on some domain Ω of the complex plane and taking values in the set of linear mappings from a linear space \mathcal{X} to a linear space \mathcal{Y} . If $x(s)$ is an analytic vector-valued function taking values in \mathcal{X} then the first r coefficients in the Taylor series development of $M(s)x(s)$ around any point $\lambda \in \Omega$ are determined by the first r coefficients in the Taylor series development of $x(s)$ around λ . The dependence is of course linear, and we denote the associated mapping by $M^{[r]}(\lambda)$, which is a linear mapping from the r -fold product \mathcal{X}^r to the r -fold product \mathcal{Y}^r . By repeating this construction at every $\lambda \in \Omega$, we obtain a new operator-valued function $M^{[r]}(s)$, which we shall call the *r -fold blow-up* of $M(s)$. An explicit expression for $M^{[r]}(s)$ in terms of $M(s)$ is given by

$$M^{[r]}(s) = \begin{bmatrix} M(s) & 0 & \cdots & \cdots & 0 \\ M'(s) & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \frac{1}{(r-1)!} M^{(r-1)}(s) & \cdots & \cdots & M'(s) & M(s) \end{bmatrix}. \quad (20)$$

This clearly shows that $M^{[r]}(s)$ will again be an analytic operator-valued function. We shall sometimes use the notation $[M(s)]^{[r]}$ instead of $M^{[r]}(s)$, in particular when $M(s)$ is a partitioned matrix, and in such cases even write $[M(s)]^{[r]}(\lambda)$ instead of $M^{[r]}(\lambda)$.

The blow-up does not commute with matrix partitioning; indeed, if A and B are linear mappings from \mathcal{X} to \mathcal{Z} and from \mathcal{Y} to \mathcal{Z} respectively then $[A \ B]^{[r]}$ is a mapping from $(\mathcal{X} \times \mathcal{Y})^r$ to \mathcal{Z}^r , but $[A^{[r]} \ B^{[r]}]$ is a mapping from $\mathcal{X} \times \mathcal{Y}^r$ to \mathcal{Z}^r . To get a proper correspondence, we need an operator from $\mathcal{X} \times \mathcal{Y}^r$ to $(\mathcal{X} \times \mathcal{Y})^r$,

that $\mathcal{K} \cap \mathcal{M}$ is a direct complement of \mathcal{P} in \mathcal{M} . For the second claim, suppose that $w \in \mathcal{C} \cap \mathcal{M}$. We can write $w = w_1 + w_2$ with $w_1 \in \mathcal{K} \cap \mathcal{M}$ and $w_2 \in \mathcal{T}$. Because $w \in \mathcal{M}$ and $w_1 \in \mathcal{M}$, it follows that $w_2 \in \mathcal{M} \cap \mathcal{T} = \{0\}$, so that $w = w_1 \in \mathcal{K}$. \square

Because of the robustness criterion that we adopted, it is of interest to see what the maximum value is of $\phi(\mathcal{P}, \mathcal{C})$ under the constraint (32).

Lemma 2.6. Let \mathcal{W} be a unitary space with given subspaces \mathcal{M} , \mathcal{P} , and \mathcal{K} . Assume that $\mathcal{P} \subset \mathcal{M}$, $\mathcal{P} \cap \mathcal{K} = \{0\}$, and $\mathcal{M} = \mathcal{P} + (\mathcal{K} \cap \mathcal{M})$. Under these conditions, we have

$$\phi(\mathcal{P}, \mathcal{C}) \leq \phi(\mathcal{C}, \mathcal{K} \cap \mathcal{M}) \tag{33}$$

for all subspaces \mathcal{C} satisfying (32), and equality is achieved for instance for $\mathcal{C} = (\mathcal{K} \cap \mathcal{M}) \oplus \mathcal{M}^\perp$.

Proof. By the assumptions, we have $\mathcal{C} \cap \mathcal{M} \subset \mathcal{K} \cap \mathcal{M}$ and $\dim \mathcal{C} \cap \mathcal{M} = \dim \mathcal{M} - \dim \mathcal{P} = \dim \mathcal{K} \cap \mathcal{M}$, so that actually $\mathcal{C} \cap \mathcal{M} = \mathcal{K} \cap \mathcal{M}$. From this, it is clear that the inequality (33) holds. The fact that equality holds for $\mathcal{C} = (\mathcal{K} \cap \mathcal{M}) \oplus \mathcal{M}^\perp$ is immediate from the definition of the minimal angle. \square

To get a formula for the upper bound appearing in (33), assume that we have *normalized* kernel and image representations for the subspace \mathcal{P} , so that

$$\mathcal{P} = \text{im } P = \text{ker } \tilde{P}, \quad P^*P = I, \quad \tilde{P}\tilde{P}^* = I. \tag{34}$$

Also take an image representation C for \mathcal{C} . Because \mathcal{C} and \mathcal{P} are complementary, the matrix $\tilde{P}C$ must be invertible, and since image representations are only determined up to right multiplication by nonsingular matrices, we may as well assume that

$$\tilde{P}C = I. \tag{35}$$

Under the assumptions we have made, the projection along \mathcal{P} onto $\mathcal{K} \cap \mathcal{M} = \mathcal{C} \cap \mathcal{M}$ in \mathcal{M} is given by $C\tilde{P}|_{\mathcal{M}}$ (cf. Lemma 2.3), and so, by (15), we have $\sin \phi(\mathcal{P}, \mathcal{K} \cap \mathcal{M}) = \|C\tilde{P}|_{\mathcal{M}}\|^{-1}$. The latter expression can be further evaluated as follows, using the fact that the matrix $[P \ P^*]$ is unitary:

$$\begin{aligned} \|C\tilde{P}|_{\mathcal{M}}\| &= \|C|_{\tilde{P}\mathcal{M}}\| = \left\| \begin{bmatrix} P^* \\ \tilde{P} \end{bmatrix} C|_{\tilde{P}\mathcal{M}} \right\| \\ &= (1 + \|P^*C|_{\tilde{P}\mathcal{M}}\|^2)^{1/2}. \end{aligned} \tag{36}$$

In all, we get (under the assumptions (34) and (35))

$$\sin \phi(\mathcal{P}, \mathcal{K} \cap \mathcal{M}) = (1 + \|P^*C|_{\tilde{P}\mathcal{M}}\|^2)^{-1/2}. \tag{37}$$

Our optimization method will be based on a

parametrization of all regulators provided by Cevik and Schumacher (1994). The key facts are as follows.

Lemma 2.7. Let $\tilde{P}(s)$ be a kernel representation of the subspace-valued function $\mathcal{P}(s)$ defined in (7), and let $\tilde{M}(s)$ be a kernel representation of the sequence of subspace-valued functions $\mathcal{M}^{(r)}(s)$ defined in (25). Then there exists a square and nonsingular RH_∞ matrix function $\tilde{H}(s)$ such that

$$\tilde{M}(s) = \tilde{H}(s)\tilde{P}(s). \tag{38}$$

Moreover, the nontrivial elementary divisors of $\tilde{H}(s)$ are the same as those of $sI - A_{22}$.

Theorem 2.8. Consider the system (1)–(3), and let $P(s)$ and $\tilde{P}(s)$ denote image and kernel representations respectively for the subspace-valued function $\mathcal{P}(s)$ associated with the plant as defined by (7). Assume that the regulator problem with internal stability is solvable, and let $C_0(s)$ be an image representation of the function $\mathcal{C}(s)$ associated as in (9) with a particular solution normalized such that $\tilde{P}(s)C_0(s) = I$. Let $\tilde{H}(s)$ be as in Lemma 2.7. Under these conditions, the general form of an image representation $C(s)$ of a solution of the regulator problem with internal stability is given by

$$C(s) = C_0(s) - P(s)\Psi(s)\tilde{H}(s), \tag{39}$$

where $\Psi(s)$ is an arbitrary element of $RH_\infty^{m \times p}$.

3. SOLUTION OF THE REGULATOR PROBLEM WITH ROBUST STABILITY

It will be convenient to always work with *normalized* image and kernel representations for the plant; that is, we shall require these representations to satisfy

$$P^*(s)P(s) = I, \quad \tilde{P}(s)\tilde{P}^*(s) = I, \tag{40}$$

where $M^*(s)$, for a real rational matrix $M(s)$, denotes $M^T(-s)$. For the controller, we shall always work with image representations $C(s)$ normalized such that $\tilde{P}(s)C(s) = I$. Owing to these normalizations, the stability margin achieved by a controller represented by $C(s)$ is

$$\sin \phi(\mathcal{P}, \mathcal{C}) = \|C\tilde{P}\|_\infty^{-1} = \|C\|_\infty^{-1}. \tag{41}$$

In terms of the parametrization (39) obtained in the previous section, we should therefore aim at minimizing $\|C_0 - P\Psi\tilde{H}\|$ over the RH_∞ matrices

$\Psi(s)$. Applying the same trick as in the derivation of (37), we can also write

$$\begin{aligned} \sin \phi(\mathcal{P}, \mathcal{C}) &= \|C_0 - P\Psi\tilde{H}\|_\infty^{-1} \\ &= (1 + \|P^*C_0 - \Psi\tilde{H}\|_\infty^2)^{-1/2}, \end{aligned} \quad (42)$$

and so we may as well minimize $\|P^*C_0 - \Psi\tilde{H}\|_\infty$.

There are two obvious lower bounds for this problem. First of all, since $\Psi(s)\tilde{H}(s)$ is an RH_∞ matrix, it follows from Nehari's theorem (cf. e.g. Francis, 1987) that a lower bound for the minimization problem is given by the norm of the Hankel operator associated with P^*C_0 :

$$\|P^*C_0 - \Psi\tilde{H}\|_\infty \geq \|\Gamma_{P^*C_0}\|. \quad (43)$$

In the case in which we have no regulation constraints and so we have a pure robust stabilization problem, we can take $\tilde{H}(s) = I$, and then the Hankel norm is an exact lower bound (McFarlane and Glover 1990). At first sight, the bound may seem to depend on the choice of the particular stabilizing compensator $C_0(s)$; however, another compensator $C(s) = C_0(s) - P(s)Q(s)$ would produce the symbol $P^*C = P^*C_0 - Q$, which differs only by an H_∞ matrix from P^*C_0 , so that the Hankel norm would not be affected. Glover and McFarlane (1989) give an expression that is explicitly independent of the choice of the compensator: they show that

$$(1 + \|\Gamma_{P^*C_0}\|^2)^{-1/2} = (1 - \|\Gamma_{\tilde{P}^*}\|^2)^{1/2}. \quad (44)$$

The second lower bound that is immediately seen to hold is

$$\begin{aligned} \|P^*C_0 - \Psi\tilde{H}\|_\infty &\geq \|P^*(\lambda)C_0(\lambda)|_{\ker \tilde{H}(\lambda)}\| \\ &\quad \forall \lambda \in i\mathbb{R} \text{ of } \tilde{H}(s). \end{aligned} \quad (45)$$

From the analysis in the previous section, we see that this inequality is really of a geometric nature. Indeed, since $\tilde{P}(\lambda)$ is surjective for all $\lambda \in i\mathbb{R}$, we have

$$\tilde{P}(\lambda) \ker \tilde{M}(\lambda) = \tilde{P}(\lambda) \ker \tilde{H}(\lambda) \tilde{P}(\lambda) = \ker \tilde{H}(\lambda),$$

so it follows from (37) that the above inequality may also be written as

$$\phi(\mathcal{P}, \mathcal{C}) \leq \phi(\mathcal{P}(\lambda), \mathcal{K} \cap \mathcal{M}(\lambda)). \quad (46)$$

Our goal in this section will be to show that the actual situation is as good as one might hope on the basis of the above two inequalities, namely that RPRS(γ) is solvable whenever RPIS is solvable and we have both

$$\gamma < (1 + \|\Gamma_{P^*C_0}\|^2)^{-1/2} \quad (47)$$

and

$$\gamma < \sin \phi(\mathcal{P}(\lambda), \mathcal{K} \cap \mathcal{M}(\lambda)) \quad \forall \lambda \in \sigma(A_{22}). \quad (48)$$

Our strategy to show this will be as follows. First, we use the well-known parametrization of

all suboptimal solutions to the Nehari problem in terms of a norm-bounded parameter in order to transform the problem into a boundary Nevanlinna–Pick problem (Ball *et al.* 1990, Chap. 21); this will require, of course, a translation of the interpolation data for the Nehari problem into interpolation constraints on the parameter. We then show that the Nevanlinna–Pick problem is solvable by proving that an associated Pick matrix is positive-definite. It should be noted that alternative approaches would be possible, for instance by adapting the method of Hara *et al.* (1992) to the case at hand. We believe that the derivation below provides a reasonably transparent route.

First we introduce some convenient notation and a rescaling. We shall write

$$R(s) = P^*(s)C_0(s), \quad (49)$$

$$W(s) = R(s) - \Psi(s)\tilde{H}(s), \quad (50)$$

so we let $W(s)$ play the role of a parameter rather than $\Psi(s)$. By a suitable rescaling, we may assume that the given bound is 1. Therefore what we need to prove can be formulated as follows (cf. Hara *et al.*, 1992, Theorem 3).

Theorem 3.1. Let $R(s)$ be a given matrix in $RL_\infty^{m \times p}$, and let $\tilde{H}(s)$ be a given nonsingular matrix in $RH_\infty^{p \times p}$ having zeros only on the finite imaginary axis. Assume that

$$\|\Gamma_R\| < 1 \quad (51)$$

and that

$$\|R(\lambda)|_{\ker \tilde{H}(\lambda)}\| < 1 \quad (52)$$

for all zeros λ of $\tilde{H}(s)$. Under these conditions, there exists a matrix $W \in RL_\infty^{m \times p}$ such that the following conditions hold:

$$W - R \in RH_\infty^{m \times p}, \quad (53)$$

$$W^{[r]}(\lambda)|_{\ker \tilde{H}^{[r]}(\lambda)} = R^{[r]}(\lambda)|_{\ker \tilde{H}^{[r]}(\lambda)} \quad (54)$$

for all zeros λ of $\tilde{H}(s)$ of multiplicity r , and

$$\|W\|_\infty \leq 1. \quad (55)$$

The proof of the theorem will proceed through a number of lemmas. In the first of these we replace the interpolation constraints (54) by a stronger version, which will be convenient below. Of course, the replacement has to be done in such a way that in particular the norm constraint (55) can still be satisfied, and this is the main point of the lemma. In the scalar case the lemma is trivial.

Lemma 3.2. In the situation of Theorem 3.1, let λ be a zero of $\tilde{H}(s)$ of multiplicity r . There exist matrices W_0, \dots, W_{r-1} with $\|W_0\| < 1$ such that

the interpolation constraint (54) holds for any matrix $W(s) \in RL_\infty^{m \times p}$ such that

$$\frac{1}{j!} W^{(j)}(\lambda) = W_j \quad (j = 0, \dots, r-1). \quad (56)$$

Proof. For this proof, we use the representation of interpolation data by means of right null chains as in Ball *et al.* (1990, Section 1.2). Choose a canonical set of right null chains for $\tilde{H}(s)$ at λ

$$x_{10}, \dots, x_{1,r_1-1}; x_{20}, \dots, x_{2,r_2-1}; \dots; x_{k0}, \dots, x_{k,r_k-1},$$

with $r_1 \geq \dots \geq r_k$. Introduce the indices $\mu_j = \max\{l \mid r_l > j\}$, and for $0 \leq i \leq j \leq r-1$ form the matrices

$$X_{ij} = [x_{1i} \ \dots \ x_{\mu_j i}].$$

A matrix function $W(s)$ will satisfy the interpolation constraints (54) if it satisfies (56) and

$$W_0 X_{00} = R(\lambda) X_{00}, \quad (57)$$

$$W_1 X_{01} + W_0 X_{11} = R'(\lambda) X_{01} + R(\lambda) X_{11}, \quad (58)$$

⋮

$$\begin{aligned} &W_{r-1} X_{0,r-1} + \dots + W_0 X_{r-1,r-1} \\ &= \frac{1}{(r-1)!} R^{(r-1)}(\lambda) X_{0,r-1} + \dots + R(\lambda) X_{r-1,r-1}. \end{aligned} \quad (59)$$

The vectors x_{10}, \dots, x_{k0} are linearly independent and span the space $\ker \tilde{H}(\lambda)$ (Ball *et al.*, 1990, Proposition 1.2.2). The matrix X_{00} is therefore a basis matrix for $\ker \tilde{H}(\lambda)$, and because of the assumption (52) we can find a matrix W_0 of norm less than one such that (57) holds. The matrices $X_{01}, \dots, X_{0,r-1}$ are of full column rank as well, and so we can solve the equations (58)–(59) recursively to get W_1, \dots, W_{r-1} . The matrices so obtained satisfy the conditions of the lemma. \square

The solution to the Nehari problem can be parametrized as

$$W = (\Theta_{11}G + \Theta_{12})(\Theta_{21}G + \Theta_{22})^{-1}, \quad (60)$$

where the matrix

$$\Theta(s) = \begin{bmatrix} \Theta_{11}(s) & \Theta_{12}(s) \\ \Theta_{21}(s) & \Theta_{22}(s) \end{bmatrix} \quad (61)$$

can be explicitly constructed from state space data for $R(s)$ (Glover, 1984; Ball and Ran, 1986, 1987). Our next step will be to give interpolation

data on the parameter G in order for W as determined by (60) to satisfy (56). We do this first without regard to the norm constraints.

Lemma 3.3. Let matrices W_0, \dots, W_{r-1} of size $m \times p$ be given, and let a matrix function $\Theta(s)$ of size $(m+p) \times (m+p)$ be given as in (61). Define matrices W^r and F^r of size $rp \times m$ by

$$W^r = \begin{bmatrix} W_0 \\ W_1 \\ \vdots \\ W_{r-1} \end{bmatrix}, \quad F^r = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Let $\lambda \in \mathbb{C}$ be given, and assume that $\Theta(\lambda)$ is invertible. Define matrices N^r and D^r by

$$\text{Mi} \begin{bmatrix} N^r \\ D^r \end{bmatrix} = [\Theta^{[r]}(\lambda)]^{-1} \text{Mi} \begin{bmatrix} W^r \\ F^r \end{bmatrix}. \quad (62)$$

If now $G(s)$ is an $m \times p$ matrix function such that $\Theta_{21}(s)G(s) + \Theta_{22}(s)$ is nonsingular and

$$G^{[r]}(\lambda)D^r = N^r, \quad (63)$$

the matrix function $W(s)$ defined by (60) satisfies

$$\frac{1}{j!} W^{(j)}(\lambda) = W_j \quad (j = 0, 1, \dots, r-1). \quad (64)$$

Proof. Equation (60) may be written in the form

$$\begin{bmatrix} W \\ I \end{bmatrix} = \Theta \begin{bmatrix} G \\ I \end{bmatrix} (\Theta_{21}G + \Theta_{22})^{-1}, \quad (65)$$

and (63) implies

$$\begin{aligned} \begin{bmatrix} G \\ I \end{bmatrix}^{[r]}(\lambda) D^r &= \text{Mi} \begin{bmatrix} G^{[r]}(\lambda) \\ I \end{bmatrix} D^r \\ &= \text{Mi} \begin{bmatrix} N^r \\ D^r \end{bmatrix}. \end{aligned} \quad (66)$$

Therefore we can rewrite (62) as

$$\begin{aligned} \text{Mi} \begin{bmatrix} W^r \\ F^r \end{bmatrix} &= \Theta^{[r]}(\lambda) \text{Mi} \begin{bmatrix} N^r \\ D^r \end{bmatrix} \\ &= \Theta^{[r]}(\lambda) \begin{bmatrix} G \\ I \end{bmatrix}^{[r]}(\lambda) D^r \\ &= \begin{bmatrix} W \\ I \end{bmatrix}^{[r]}(\lambda) (\Theta_{21}G + \Theta_{22})^{[r]}(\lambda) D^r \\ &= \text{Mi} \begin{bmatrix} W^{[r]}(\lambda) \\ I \end{bmatrix} (\Theta_{21}G + \Theta_{22})^{[r]}(\lambda) D^r. \end{aligned} \quad (67)$$

From the resulting equations

$$W^r = W^{[r]}(\lambda) (\Theta_{21}G + \Theta_{22})^{[r]}(\lambda) D^r, \quad (69)$$

$$F^r = (\Theta_{21}G + \Theta_{22})^{[r]}(\lambda) D^r, \quad (70)$$

we immediately get

$$W^r = W^{[r]}(\lambda)F^r, \tag{71}$$

which is the same as (64). \square

In connection with the norm constraint in the parametrization of the suboptimal solutions of the Nehari problem, the matrix $\Theta(s)$ is required to be J -unitary (i.e. $\Theta^*J\Theta = J$), where

$$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

In the next lemma we can translate the assumption $\|W_0\| < 1$ on the interpolation data for the parameter W to an assumption on the interpolation data for the parameter G . The argument in the proof is a standard one.

Lemma 3.4. In the situation of the previous lemma, suppose that $\Theta(s)$ is J -unitary and that $\|W\| < 1$. We then have

$$N_0^*N_0 < D_0^*D_0, \tag{72}$$

where N_0 and D_0 denote the upper blocks in the matrices N^r and D^r respectively.

Proof. From the definition (62), it follows that in particular

$$\begin{bmatrix} W_0 \\ I \end{bmatrix} = \Theta(\lambda) \begin{bmatrix} N_0 \\ D_0 \end{bmatrix}, \tag{73}$$

so that

$$\begin{aligned} N_0^*N_0 - D_0^*D_0 &= \begin{bmatrix} N_0 \\ D_0 \end{bmatrix}^* J \begin{bmatrix} N_0 \\ D_0 \end{bmatrix} \\ &= \begin{bmatrix} N_0 \\ D_0 \end{bmatrix}^* \Theta^*(\lambda) J \Theta(\lambda) \begin{bmatrix} N_0 \\ D_0 \end{bmatrix} \\ &= \begin{bmatrix} W_0 \\ I \end{bmatrix}^* J \begin{bmatrix} W_0 \\ I \end{bmatrix} \\ &= W_0^*W_0 - I < 0. \end{aligned} \tag{74}$$

\square

The suboptimal solutions of the Nehari problem are obtained by using a parameter G in (60) that satisfies the norm constraint $\|G\|_\infty < 1$. After the reformulations of the preceding lemmas, we see that we can get a solution of the original problem if we can find a matrix function $G(s)$ in $RH_\infty^{m \times p}$ such that $\|G\|_\infty < 1$ and G satisfies the interpolation constraints (63) at a number of points λ on the imaginary axis, where

we may assume that (72) holds. This is the *boundary Nevanlinna-Pick* (NP) problem. The fact that the boundary NP problem comes up in connection with regulation constraints in an H_∞ context has been recognized before by Sugie and Hara (1989). The following lemma can be seen as an extension of their Lemma B.

Lemma 3.5. Let there be given numbers λ_i ($i = 1, \dots, n$), all on the imaginary axis, and matrices

$$D_i = \begin{bmatrix} D_{i0} \\ D_{i1} \\ \vdots \\ D_{i,r-1} \end{bmatrix}, \quad N_i = \begin{bmatrix} N_{i0} \\ N_{i1} \\ \vdots \\ N_{i,r-1} \end{bmatrix}, \tag{75}$$

$i = 1, \dots, n.$

There exists an RH_∞ matrix $G(s)$ such that $\|G\|_\infty < 1$ and

$$G^{[r]}(\lambda_i)D_i = N_i \quad (i = 1, \dots, n) \tag{76}$$

if and only if

$$N_{i0}^*N_{i0} < D_{i0}^*D_{i0} \quad (i = 1, \dots, n). \tag{77}$$

Proof. We only sketch the proof here; a more detailed proof is provided in the Appendix. Consider the (more demanding) problem of finding a function $G(s)$ that is analytic and less than one in modulus on a region $\{s \in \mathbb{C} \mid \text{Re } s \geq \varepsilon\}$ ($\varepsilon > 0$) and that satisfies the interpolation constraints. This problem is no longer a *boundary* NP problem, since the interpolation points are now inside the region of analyticity, and so one can form the Pick matrix, which of course depends on ε . The problem is solvable if and only if this Pick matrix, which we shall denote by $P(\varepsilon)$, is positive-definite. Upon examining the behavior of the elements of $P(\varepsilon)$ as ε tends to zero, one finds that the diagonal elements tend to $+\infty$, whereas the off-diagonal elements remain bounded. Therefore $P(\varepsilon)$ is guaranteed to be positive-definite for sufficiently small ε , and the problem is solved. \square

Putting all the lemmas together, it is now easy to get a proof of the theorem.

Proof of Theorem 3.1. For each zero λ of $\tilde{H}(s)$ of multiplicity r , construct matrices W_0, \dots, W_{r-1} as in Lemma 3.2. From these, construct interpolation data for the parameter $G(s)$ as in Lemma 3.3. It follows from Lemmas 3.5 and 3.4 that these interpolation data can be satisfied by an RH_∞ matrix $G(s)$ of H_∞ norm less than one. By the parametrization of solutions to the

suboptimal Nehari problem (see e.g. Ball and Ran, 1987), the matrix $W(s)$ given by (60) satisfies the conditions of the theorem. \square

This leads to the main result of the paper.

Theorem 3.6. Consider the problems RPIS, RSP(γ) and RPRS(γ), as described in Section 2. Define subspace-valued functions $\mathcal{P}(s)$ and $\mathcal{M}(s)$ by (7) and (8) respectively, and define \mathcal{H} by (26). Under assumptions (A1)–(A4), the problem RPRS(γ) is solvable if and only if the following conditions hold:

- (i) RPIS is solvable;
- (ii) RSP(γ) is solvable;
- (iii) $\gamma < \sin \phi(\mathcal{P}(\lambda), \mathcal{H} \cap \mathcal{M}(\lambda))$ for all exosystem poles λ .

Proof. The necessity of conditions (i) and (ii) is obvious from the problem formulation, and the necessity of (iii) follows from Theorem 2.2 and Lemma 2.6. Assume now that (i)–(iii) hold. Let $P(s)$ and $\tilde{P}(s)$ denote image and kernel representations of the function $\mathcal{P}(s)$, normalized as in (40), and let $C_0(s)$ be an image representation of a particular solution of RPIS, normalized such that $\tilde{P}(s)C_0(s) = I$. Construct $\tilde{M}(s)$ as in the proof of Lemma 2.7, and compute $\tilde{H}(s)$ such that $\tilde{M}(s) = \tilde{H}(s)\tilde{P}(s)$. Note that

$$\tilde{P}(\lambda)\mathcal{M}(\lambda) = \tilde{P}(\lambda) \ker \tilde{H}(\lambda)\tilde{P}(\lambda) = \ker \tilde{H}(\lambda) \quad (78)$$

for all $\lambda \in \mathbb{C}^+$, because $\tilde{P}(\lambda)$ is surjective for all such λ by the stabilizability and detectability assumptions. Define $\alpha = \gamma^{-1}\sqrt{1-\gamma^2}$, and write $R(s) = \alpha^{-1}P^*(s)C_0(s)$. It follows from (iii) with (37) that $\|R(\lambda)|_{\ker \tilde{H}(\lambda)}\| < 1$ for all exosystem poles λ . Also, (ii) implies that $\|\Gamma_R\| < 1$. Compute the matrix $\Theta(s)$ in (61) as indicated for instance in Ball and Ran (1987). For each zero λ of $\tilde{H}(s)$ of multiplicity r , compute matrices W_0, \dots, W_{r-1} as in the proof of Lemma 3.2, and from these compute interpolation data (D_0, \dots, D_{r-1}) and (N_0, \dots, N_{r-1}) as in Lemma 3.3. Collecting all the data from the various zeros of $\tilde{H}(s)$, compute a matrix $G(s)$ as in Lemma 3.5. Next, find $W(s)$ from $G(s)$ by (60). The matrix $W(s)$ will then satisfy the conditions (53)–(55), and it follows that $C(s) = C_0(s) + \alpha P(s)[W(s) - R(s)]$ provides an image representation of a solution of RPRS(γ). \square

The sufficiency part of the proof is constructive. State-space parameters for the compensator can be obtained from an image representation as for instance in Fuhrmann (1981). In the next section we illustrate the computational procedure with an example.

4. EXAMPLE

To illustrate the methods of this paper in the simplest possible context, let us consider a first-order system that is to be regulated against a constant disturbance. After scaling, the equations can be written in the form

$$\dot{x}_1 = ax_1 + x_2 + u, \quad (79)$$

$$\dot{x}_2 = 0, \quad (80)$$

$$y = x_1. \quad (81)$$

Assumptions (A1)–(A4) are satisfied for all values of a . The subspace-valued function $\mathcal{M}(s)$ defined in (8) is given by

$$\mathcal{M}(s) = \ker [s(s-a) \quad -s]. \quad (82)$$

This expression can be obtained symbolically by eliminating x_1 and x_2 from the equations $(s-a)x_1 = x_2 + u$, $sx_2 = 0$ and $y = x_1$, considering s as a noncancellable parameter. If we do allow cancellation, we get the expression for $\mathcal{P}(s)$ as in (7):

$$\mathcal{P}(s) = \ker [s-a \quad -1]. \quad (83)$$

The upper bound on the achievable robustness of stability imposed by the regulation constraint is obtained from (46), noting that $\mathcal{M}(0) = \mathbb{C}^2 = \mathcal{Y} \times \mathcal{U}$ and $\mathcal{H} = \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

$$\begin{aligned} \sin \phi(\mathcal{P}(0), \mathcal{H}) &= \sin \phi \left(\text{im} \begin{bmatrix} -1 \\ a \end{bmatrix}, \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= \frac{1}{\sqrt{1+a^2}}. \end{aligned} \quad (84)$$

The achievable robustness of stability, not taking into account the regulation constraint, can be computed from either side of (44). In view of the fact that a particular stabilizing controller $C_0(s)$ will have to be computed anyway, there is perhaps no clear preference for either method of computation. First we need normalized image and kernel representations for $\mathcal{P}(s)$; these are given by

$$P(s) = \frac{1}{s + \sqrt{a^2 + 1}} \begin{bmatrix} 1 \\ s-a \end{bmatrix}, \quad (85)$$

$$\tilde{P}(s) = \frac{1}{s + \sqrt{a^2 + 1}} [s-a \quad -1]. \quad (86)$$

An image representation $C_0(s)$ of a stabilizing controller, normalized such that $\tilde{P}(s)C_0(s) = I$, is found by solving the equation

$$(s-a)c_{01}(s) - c_{02}(s) = s + \sqrt{a^2 + 1} \quad (87)$$

in the unknown RH_∞ functions $c_{01}(s)$ and $c_{02}(s)$. A simple solution is provided by

$$C_0(s) = \begin{bmatrix} 1 \\ -a - \sqrt{a^2 + 1} \end{bmatrix}. \tag{88}$$

We get

$$\begin{aligned} (1 + \|\Gamma_{P^*C_0}\|^2)^{-1/2} &= (1 - \|\Gamma_{\tilde{P}^*}\|^2)^{1/2} \\ &= \frac{1}{2}\sqrt{2} \left(1 - \frac{a}{\sqrt{a^2 + 1}}\right)^{1/2}. \end{aligned} \tag{89}$$

A somewhat more attractive formulation is obtained if we reparametrize by setting

$$a = \cot \theta, \quad 0 < \theta < \pi. \tag{90}$$

Note that $a \rightarrow \infty$ as $\theta \downarrow 0$, and $a \rightarrow -\infty$ as $\theta \uparrow \pi$. After some trigonometry, we find for the two upper bounds

$$\sin \phi(\mathcal{P}(0), \mathcal{K}) = \sin \theta, \tag{91}$$

$$(1 + \|\Gamma_{P^*C_0}\|^2)^{-1/2} = \sin \frac{1}{2}\theta. \tag{92}$$

Note that $\sin \theta > \sin \frac{1}{2}\theta$ for $0 < \theta < \frac{2}{3}\pi$; the value $\theta = \frac{2}{3}\pi$ corresponds to $a = -\frac{1}{3}\sqrt{3}$. Taking into account that $-a^{-1}$ is the d.c. gain of the system (79) and that this quantity is not affected by the scaling we used to obtain the form (79), we arrive at the following conclusion:

for a first-order system regulated against a constant disturbance, the regulation requirement is restrictive with respect to the achievable robustness of stability if and only if the system is open-loop stable with a d.c. gain less than $\sqrt{3}$.

Of course, similar rules may be derived for higher-order systems and for other types of regulation constraints.

Let us now proceed to the calculation of an actual controller. In order to simplify matters even further, we shall from now on assume that the parameter a in (4.79) is equal to zero. According to the rule given above, the regulation requirement is in this case not restrictive with respect to robustness of stability, and so we should be able to get arbitrarily close to the optimal margin of stability while at the same time achieving regulation against constant disturbances. Specifically, the upper bound given by (84) is 1, whereas the one given by (89) is $\frac{1}{2}\sqrt{2}$. We shall compute a controller that achieves a robustness margin of at least γ , where γ is a given number less than $\frac{1}{2}\sqrt{2}$, and that at the same time satisfies the regulation requirement.

Taking the particular stabilizing controller $C_0(s)$ of (88), we get from the Kučera–Youla parametrization (Kučera, 1974; Youla *et al.*,

1976) the following general form for an image representation of a stabilizing controller:

$$C(s) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{Q(s)}{s+1} \begin{bmatrix} 1 \\ s \end{bmatrix}, \tag{93}$$

where $Q(s)$ is an arbitrary RH_∞ function. The regulation requirement (28) is in the present case

$$\mathcal{C}(0) \subset \text{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{94}$$

which will be satisfied for a representation of the form (93) if and only if $Q(0) = 1$. Therefore a particular solution to RPIS is given (with a change of notation) by

$$C_0(s) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \frac{1}{s+1} \begin{bmatrix} 1 \\ s \end{bmatrix} = \frac{1}{s+1} \begin{bmatrix} s \\ -(2s+1) \end{bmatrix}, \tag{95}$$

and the general solution to RPIS is

$$C(s) = \frac{1}{s+1} \begin{bmatrix} s \\ -(2s+1) \end{bmatrix} - \frac{s\Psi(s)}{(s+1)^2} \begin{bmatrix} 1 \\ s \end{bmatrix}, \tag{96}$$

where $\Psi(s)$ is an arbitrary RH_∞ function. This is in line with the result in Theorem 2.8; note that the function $\tilde{H}(s)$ can in the present case be chosen as $\tilde{H}(s) = s/(s+1)$. Now, define

$$\alpha = \frac{1}{\gamma} \sqrt{1 - \gamma^2} > 1, \tag{97}$$

and write

$$R(s) = \frac{1}{\alpha} P^*(s) C_0(s) = \frac{1}{\alpha} \frac{2s}{1-s} = \frac{2}{\alpha} \frac{1}{1-s} - \frac{2}{\alpha}. \tag{98}$$

We are now looking for a Nehari extension $W(s)$ of $R(s)$ that satisfies the norm bound $\|W\|_\infty \leq 1$ and the interpolation constraint (54), which in this case comes down to

$$W(0) = 0. \tag{99}$$

All extensions satisfying the norm constraint are given by

$$\begin{aligned} W(s) &= [\Theta_{11}(s)G(s) + \Theta_{12}(s)] \\ &\quad \times [\Theta_{21}(s)G(s) + \Theta_{22}(s)]^{-1}, \end{aligned} \tag{100}$$

where $G(s)$ is an arbitrary RH_∞ function of norm less than 1 and where the matrix $\Theta(s)$ is computed from the formulas in Ball and Ran (1987):

$$\begin{aligned} \Theta(s) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{2}{(\alpha^2 - 1)(s^2 - 1)} \\ &\quad \times \begin{bmatrix} -(s+1) & -\alpha(s+1) \\ \alpha(s-1) & s-1 \end{bmatrix}. \end{aligned} \tag{101}$$

The next step is to translate the interpolation constraint on $W(s)$ into one on the parameter $G(s)$. The general procedure for doing this is given by Lemma 3.3; in the present case we obtain from

$$\begin{bmatrix} W(0) \\ 1 \end{bmatrix} = \Theta(0) \begin{bmatrix} G(0) \\ 1 \end{bmatrix} \quad (102)$$

the constraint

$$G(0) = \frac{-2\alpha}{\alpha^2 + 1}. \quad (103)$$

Now we have to solve the boundary Nevanlinna–Pick problem of finding an RH_∞ function of norm less than one that satisfies (103). This is not at all difficult: we can simply take the constant solution

$$G(s) = \frac{-2\alpha}{\alpha^2 + 1}. \quad (104)$$

From this, we get

$$W(s) = \frac{s+1}{s-1} \frac{-2\alpha s}{(\alpha^2 + 1)s + \alpha^2 - 1}, \quad (105)$$

which gives

$$R(s) - W(s) = \frac{2}{\alpha} \frac{s}{(\alpha^2 + 1)s + \alpha^2 - 1}. \quad (106)$$

The result is in RH_∞ and is a multiple of s , as it should be. The parameter $\Psi(s)$ in (96) becomes (see (50), and take the scaling by α into account)

$$\begin{aligned} \Psi(s) &= \alpha \frac{s+1}{s} [R(s) - W(s)] \\ &= \frac{2(s+1)}{(\alpha^2 + 1)s + \alpha^2 - 1}. \end{aligned} \quad (107)$$

Inserting this into (96), we obtain

$$C(s) = \frac{1}{(\alpha^2 + 1)s + \alpha^2 - 1} \begin{bmatrix} (\alpha^2 + 1)s \\ -2\alpha^2 s - \alpha^2 + 1 \end{bmatrix}. \quad (108)$$

This is our final solution. The controller transfer function is given by

$$c(s) = \frac{-2\alpha^2 s - \alpha^2 + 1}{(\alpha^2 + 1)s} \quad (109)$$

and has a pole at 0, as it should to satisfy the regulation requirement. The actual margin of robustness of stability achieved by the above controller is

$$\sin \phi(\mathcal{P}, \mathcal{C}) = \left[1 + \left(\frac{2\alpha}{\alpha^2 + 1} \right)^2 \alpha^2 \right]^{-1/2}, \quad (110)$$

and this is indeed better than the required margin $\gamma = (1 + \alpha^2)^{-1/2}$. If we let γ tend to its optimal value $\frac{1}{2}\sqrt{2}$ then α tends to 1 and the controller tends to $C(s) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; this controller

optimizes the robustness margin, but it no longer satisfies the regulation constraint.

5. CONCLUSIONS

The problem of optimizing the robustness of stability with respect to coprime factor perturbations was posed by Vidysagar and Kimura (1986) and was reduced in that paper to a certain H_∞ optimization problem. Later on, it was shown by Glover and McFarlane (1989) that if the perturbations are taken with respect to *normalized* coprime factors then an exact solution can be obtained in a relatively simple way. The fact that the optimal robust stabilization problem is such a nice one came as a surprise at the time. In this paper we have shown that the problem even remains nice if we add regulation constraints to it; in view of the generally adverse behavior of optimization problems when side constraints are added, this outcome may be viewed as a new surprise.

Our techniques in this paper have relied in particular on the use of subspace-valued functions associated with both plant and controller. These can be seen as a multivariable generalization of the Nyquist curve, as may be argued in a mathematical sense using the identification of the extended complex plane with the Grassmannian $G^1(\mathbb{C}^2)$. This paper, however, has demonstrated more than that: we have shown that the multivariable Nyquist curve continues to play the role of a mediator between various design objectives, as the scalar version does in classical control theory.

One modern approach towards integration of various design objectives is the loop shaping approach developed by Glover and co-workers (McFarlane and Glover, 1990), which has already seen several successful applications, in particular to controller design for aircraft (see also Hyde and Glover, 1993; Postlethwaite and Skogestad, 1993). In this approach the construction of robustly stabilizing controllers is used as the basic synthesis procedure. The present paper has shown when and how it is possible to incorporate regulation requirements in this design method.

A noticeable difference between the scalar and the multivariable versions of the Nyquist curve is that geometry plays a much larger role in the latter than in the former. The reason for this is that the scalar version can be interpreted as a

function whose values are one-dimensional subspaces of a two-dimensional space, and not much subspace geometry is possible in two dimensions; in particular there are no subspace inclusions that are not trivial in one way or another. In the multivariable case, however, we do have a nontrivial setting, leading to formulations that would seem contrived in the scalar case. The role of finite-dimensional geometry is clear for instance in the characterization that we gave of the upper bound on the achievable robustness of stability due to regulation requirements.

A natural question to ask is whether it is possible to include robustness of regulation (Wonham, 1979, Chap. 8; Vidyasagar, 1985, Section 7.5) along with robustness of stability. In forthcoming work it will be shown that robustness of regulation can indeed be incorporated by an appropriate sharpening of the interpolation conditions, and that the trade-off against robustness of stability can be assessed in much the same way as in the present paper.

Among our standing assumptions (A1)–(A4), the full column rank assumption (A4) is the most restrictive. If this assumption does not hold, the set of compensators that achieve regulation with internal stability is no longer completely described by (39). An example of such a situation occurs in systems with hydraulic actuators, which ensure that the plant already has a zero at the origin of the complex plane, so that tracking of step inputs is automatically guaranteed. Another extension of the present work that would be of interest is to consider the case in which the to-be-controlled outputs are not necessarily the same as the observed outputs.

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APPENDIX—PROOF OF LEMMA 3.5

The main point of this proof is to show that the diagonal blocks in the Pick matrix, which are mentioned in the sketch of the proof given in the main text, tend to infinity in the appropriate sense.

The necessity of (77) is easy to see: since, in particular, $G(\lambda_i)D_{i0} = N_{i0}$, we have $\|N_{i0}x\| = \|G(\lambda_i)D_{i0}x\| < \|D_{i0}x\|$ for all x , which is the same as (77). To prove sufficiency, we turn the problem into an ordinary NP problem as described above for the scalar case. To allow application of the standard formulas, we shift the interpolation points to the right rather than the boundary to the left. Following Ball *et al.* (1990, Chap. 18), we introduce

$$A_i(\epsilon) = (sI)^{r_i}(\lambda_i + \frac{1}{2}\epsilon), \quad A = \text{diag}(A_1, \dots, A_n), \quad (\text{A.1})$$

$$C_i = [D_{i,r_i-1} \dots D_{i0}], \quad C = [C_1 \dots C_n], \quad (\text{A.2})$$

$$Z_i = [N_{i,r_i-1} \dots N_{i0}], \quad Z = [Z_1 \dots Z_n]. \quad (\text{A.3})$$

The shifted interpolation problem can now be formulated as follows: find an RH_∞ matrix $G(s)$ with $\|G\|_\infty < 1$ such that

$$\sum_{\lambda \in \sigma(A(\epsilon))} \text{Res } G(s)C[sI - A(\epsilon)]^{-1} = Z. \quad (\text{A.4})$$

The Pick matrix associated with this interpolation problem is the solution $P(\epsilon)$ of the Lyapunov equation

$$P(\epsilon)A(\epsilon) + A^*(\epsilon)P(\epsilon) = C^*C - Z^*Z \quad (\text{A.5})$$

(Ball *et al.*, 1990, Theorem 18.5.1). Writing this out in blocks corresponding to the block structure of $A(\epsilon)$, we get

$$P_{ij}(\epsilon)A_j(\epsilon) + A_i^*(\epsilon)P_{ij}(\epsilon) = C_i^*C_j - Z_i^*Z_j \quad (i, j = 1, \dots, n). \quad (\text{A.6})$$

As a consequence of the block-diagonal structure of $A(\epsilon)$, these equations are decoupled and can be solved separately. For $i \neq j$ (the off-diagonal blocks), the eigenvalues of $A_j(\epsilon)$ and $-A_i^*(\epsilon)$ are distinct for all ϵ , and so the solution $P_{ij}(\epsilon)$ of (A.6) tends to a finite limit as ϵ tends to zero (Gantmacher, 1959; VIII.3). We claim that for the diagonal blocks there exists a positive constant c such that

$$P_{ii}(\epsilon) \geq \frac{c}{\epsilon} I \quad (\text{A.7})$$

for all sufficiently small ϵ . If this holds then we can write

$$P(\epsilon) \geq \frac{1}{\epsilon} [cI + \epsilon \hat{P}(\epsilon)], \quad (\text{A.8})$$

where $\hat{P}(\epsilon)$ tends to a finite limit as ϵ tends to zero; surely then, the expression in square brackets will be positive-definite for sufficiently small ϵ , so that $P(\epsilon)$ will then be positive-definite and the problem is solved. An explicit expression for the interpolants is given in Ball *et al.* (1990, Theorem 18.5.1). (Note that it is not sufficient to show that $\|P_{ii}(\epsilon)\|$ tends to infinity, as suggested in the proof of Lemma B in Sugié and Hara (1989); it is required that *all* eigenvalues of $P_{ii}(\epsilon)$ tend to infinity, not just the largest one. Of course, this difficulty does not occur in the scalar case (Ball *et al.*, 1990, Section 21.1; 1991, p. 154).)

It remains to verify the claim (A.7). Since the problems for

different values of the index are decoupled, we may as well drop the index and consider just one interpolation point. Moreover, since the imaginary part of the point λ drops out of the equation (A.6) for $j = i$, there is no loss of generality in assuming that this interpolation point is the origin of the complex plane. In order to avoid unwieldy notation, we shall present the proof for the case where the multiplicity of the interpolation r is equal to 2; the argument that we indicate, however, is valid in the general case.

The situation now comes down to the following. For positive ϵ , define a Hermitian matrix $P(\epsilon)$ by

$$P(\epsilon)(N + \frac{1}{2}\epsilon I) + (N + \frac{1}{2}\epsilon I)^*P(\epsilon) = M, \quad (\text{A.9})$$

where

$$N = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \quad (\text{A.10})$$

$$M = M^* = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad (\text{A.11})$$

with $M_{22} > 0$ (owing to (77) and the definitions (A.2) and (A.3)). We want to show that $P(\epsilon)$ satisfies an estimate of the form (A.7). The solution $P(\epsilon)$ of (A.9) can be written explicitly as

$$P(\epsilon) = \begin{bmatrix} 2\epsilon^{-3}M_{22} - \epsilon^{-2}(M_{12} + M_{21}) + \epsilon^{-1}M_{11} & -\epsilon^{-2}M_{22} + \epsilon^{-1}M_{12} \\ -\epsilon^{-2}M_{22} + \epsilon^{-1}M_{21} & \epsilon^{-1}M_{22} \end{bmatrix}. \quad (\text{A.12})$$

The function $\epsilon^3P(\epsilon)$ is a Hermitian matrix-valued function depending analytically on ϵ . Therefore it follows from Rellich's theorem (see e.g. Gohberg *et al.*, 1982, Theorem S6.3) that there exists a basis of normalized eigenvectors of $P(\epsilon)$ depending analytically on ϵ . Since all singular values of $P(\epsilon)$ can be expressed in the form $x^*P(\epsilon)x$ for some normalized eigenvector x , it will therefore be sufficient if we can prove that there exists a positive constant c such that

$$\lim_{\epsilon \downarrow 0} x(\epsilon)^*\epsilon P(\epsilon)x(\epsilon) \geq c \quad (\text{A.13})$$

for all *analytic* vector-valued functions $x(\epsilon)$ such that $\|x(\epsilon)\| = 1$. Take such an $x(\epsilon)$, and write

$$x(\epsilon) = \begin{bmatrix} x_1(\epsilon) \\ x_2(\epsilon) \end{bmatrix} = \begin{bmatrix} x_{10} + x_{11}\epsilon + x_{12}\epsilon^2 + \dots \\ x_{20} + x_{21}\epsilon + x_{22}\epsilon^2 + \dots \end{bmatrix}. \quad (\text{A.14})$$

If $x_{10} \neq 0$, the leading term in the Laurent series expansion of $\epsilon x(\epsilon)^*P(\epsilon)x(\epsilon)$ around 0 is $2\epsilon^{-2}x_{10}^*M_{22}x_{10}$, where $x_{10}^*M_{22}x_{10} > 0$ because M_{22} is positive-definite. In this case the limit on the left-hand side of (A.13) will be $+\infty$, and so (A.13) certainly holds. So now assume that $x_{10} = 0$. Note that this implies that $\|x_{20}\| = 1$, because we must have $\|x(0)\| = 1$. It can be seen from (A.12) that the constant term is now the leading one. This term equals

$$\begin{bmatrix} x_{11} \\ x_{20} \end{bmatrix}^* \begin{bmatrix} 2M_{22} & -M_{22} \\ -M_{22} & M_{22} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{20} \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} x_{11} \\ x_{20} \end{bmatrix}^* P' \begin{bmatrix} x_{11} \\ x_{20} \end{bmatrix}. \quad (\text{A.15})$$

The matrix P' appearing here is strictly positive-definite, since it is the solution of the Lyapunov equation (A.9) with $\epsilon = 1$ and M replaced by

$$\begin{bmatrix} 0 & 0 \\ 0 & M_{22} \end{bmatrix} = L^*L$$

where $L = [0 \ M_{22}^{1/2}]$; note that the pair $(L, -N - \frac{1}{2}I)$ is observable, and of course $-N - \frac{1}{2}I$ is stable. Moreover, we know that $\left\| \begin{bmatrix} x_{11} \\ x_{20} \end{bmatrix} \right\| \geq 1$, because $\|x_{20}\| = 1$, and so the expression in (A.15) is at least equal to the smallest singular value of P' , which is positive. This completes the proof. \square